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Paper - I, Unit - I

(Real Analysis)

Denseness property of real no.–

Between any two distinct real numbers there always lies a rational no. and, therefore, infinitely many rational numbers.

proof- let x, y be any two distinct real no.

$x > 0$ and $x < y \Rightarrow y - x > 0$

Then by archimedean property $\exists$ a natural number 'n' such that –

$n(y-x) > 1$ $\{$ put $x = \frac{1}{q}$ where $q \in \mathbb{R}$ in property ② $\}$

$\Rightarrow ny - nx > 1$

$\Rightarrow ny > nx + 1$ —①

for any natural no. 'm' we write –

$m > nx$ $\{\because nx$ is also a real no. so using Archimedean property $\}$

Let p be the smallest such natural number such that

$nx < p$

If $p = 1$ so that $p - 1 = 0$

we have

$p - 1 = 0 < nx < 1 = p$

$\Rightarrow p - 1 < nx < p$

If $p \geq 2$ so that $p - 1$ is a natural no.,

then we know $p - 1 < nx$

Now $nx < p = (p-1) + 1 < nx + n(y-x) = ny$

$\Rightarrow nx < p < ny \Rightarrow x < p/n < y$

since p & n are natural no. ~~&~~ ~~integers~~

also $n \neq 0$

let $p/n = r$

$\Rightarrow$ $\boxed{x < r < y}$

thus $\exists$ a rational no. r lies between x & y. Repeating the above process for x, r & y we get new rational no. as $r_1, r_2 \ldots$ such that

$$x < r_1 < r \quad \& \quad r < r_2 < y$$

continuing this process, we get infinite many rational no. between two distinct real no. x & y.

Ques- find the supremum and infimum of the following sets-

(i) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ (iii) $\left\{(-1)^n : n \in \mathbb{N}\right\}$

(ii) $\left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

Sol^n (i) Let $X = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$

$\Rightarrow X = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

The smallest element of X is $1/2$ which is lower bound of X

since $1/2 \leq x \quad \forall x \in X$

i.e every real no. less than $1/2$ is treated as the lower bounds of X .

$\Rightarrow \frac{1}{2}$ is the greatest lower bound of X i.e infimum of X

$\Rightarrow$ g.l.b $= 1/2$

But X does not have an upper bound.

(ii) Let $X = \{1 + \frac{(-1)^n}{n} \;\forall\; n \in \mathbb{N}\}$

$\Rightarrow X = \{1 + \frac{(-1)}{1}, 1 + \frac{(-1)^2}{2}, 1 + \frac{(-1)^3}{3}, 1 + \frac{(-1)^4}{4} \dots\}$

$= \{1-1, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4} \dots\}$

$= \{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7} \dots\}$

$X = \{0, 1.3, 0.6, 1.25, 0.8, 1.16, 0.83 \dots\}$

. The smallest element of X is 0 which is lower bound of X

since $0 \leq x \;\forall\; x \in X$

i.e every real no. less than 0 is treated as the lower bound of X

$\Rightarrow$ 0 is the greatest lower bound of X i.e infimum of X

$\Rightarrow$ g.l.b $= 0$

Now we have

the largest element of X is $\frac{3}{2}$ which is upper bound of X

since $\frac{3}{2} \geq x \;\forall\; x \in X$

i.e every real no. more than $\frac{3}{2}$ is treated as the upper bound of X

$\Rightarrow$ $\frac{3}{2}$ is the least upper bound

i.e supremum of X

$\Rightarrow$ $l.u.b = \frac{3}{2}$

(iii) Let $X = \{(-1)^n \ \forall \ n \in \mathbb{N}\}$

$\Rightarrow X = \{-1, 1, -1, 1, -1, 1 \ldots\}$

clearly (-1) is the infimum and 1 is the supremum of the given set.

Equipotent sets :-

Two sets A and B are equipotent or equivalent iff there exist a one-one, onto mapping from A to B.

The equipotent relation is an equivalance relation i.e it is reflexive, symmetric and transitive. It is denoted by

$A \sim B$

For ex-

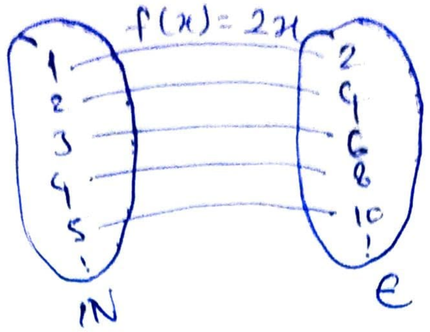
$\mathbb{N} = \{1, 2, 3 \ldots n \ldots\}$ be the set of natural no. and $E = \{2, 4, 6, 8 \ldots 2n \ldots\}$ be the set of all even no.

clearly $E \subseteq \mathbb{N}$

$\exists$ a mapping $f(x) = 2x \ \forall \ x \in \mathbb{N}$ between $\mathbb{N}$ & E

Clearly this is a one-one mapping from $\mathbb{N}$ onto E

thus the set $\mathbb{N}$ & E are equipotent sets



Countable and Uncountable sets :-

A set which is either finite or denumerable is called countable set & any set which is not countable is called uncountable set.

Denumerable OR countably infinite :-

A set 'A' is said to be denumerable if there exists a one-one mapping from the set $\mathbb{N}$ of all natural no. onto the set A i.e if $A \sim \mathbb{N}$

Finite and infinite set -

A set is called finite and is said to contain n elements if $A \sim \{1, 2, 3, \ldots n\}$. Of a set is not finite, then it is said to be infinite.

If $A \sim \{1, 2, 3, 4 \ldots n\}$, then n is called the cardinal number of A.

Theorem - The countable union of countable sets is countable.

OR

If $A_1, A_2 \ldots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

proof :- Let $\{A_i, i \in \mathbb{N}\}$ be a countable collection of countable sets. Since $A_1, A_2, A_3 \ldots$ are countable, they can be represented as follows -

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots a_{1n}, \ldots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23} \ldots a_{2n} \ldots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33} \ldots, a_{3n} \ldots\}$$
$$\ldots \ldots \ldots$$
$$\ldots \ldots \ldots$$
$$A_n = \{a_{n1}, a_{n2}, a_{n3} \ldots a_{nn} \ldots\}$$

Then a_{ij} is the jth element of A_i. We observe that there is only one element a_{11} the sum of whose suffices is 2. There are only two elements a_{12}, a_{21} the sum of whose suffices is 3. There are three elements a_{13}, a_{22}, a_{31}, the sum of whose suffices is 4. In this manner we find that for an positive integer

$m \geq 2$, there are $(m-1)$ elements, sum of whose suffices is m.

Therefore we can arrange the elements of $A_1, A_2, A_3, \dots$ according to their sum of the suffices as

$$a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, \dots,$$ where we have removed the elements a_{ij} which has already occured. Thus all the elements can be counted out and hence $\bigcup_{n=1}^{\infty} A_n$ is countable

Pictorially, the above counting process can be represented as follows –

$$\begin{array}{ccccc}
a_{11} \rightarrow & a_{12} & a_{13} \rightarrow & a_{14} & \dots \\
a_{21} & a_{22} & a_{23} & a_{24} & \dots \\
\downarrow & & & & \\
a_{31} & a_{32} & a_{33} & a_{34} & \dots \\
a_{41} & a_{42} & a_{43} & a_{44} & \dots \\
\downarrow & & & & \\
\dots & \dots & \dots & \dots & \dots
\end{array}$$